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On Exceptional Sets of Asymptotic Relations for General Orthogonal Polynomials

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The n-th root asymptotic behavior of orthonormal polynomials $q_n(z)$ corresponding to an arbitrary measure in the complex plane is studied. In particular, the following theorems are established (here Ω denotes the outer domain of the support of the measure, and $g_\Omega(z)$ is the Green function of Ω):

THEOREM 1. $\overline{\lim}_{n\to\infty} |q_n(z)|^{1/n} \geqslant e^{g_\Omega(z)}$ *everywhere in* Ω.

THEOREM 2. $\overline{\lim}_{n\to\infty} |q_n(z)|^{1/n} \geqslant 1$ *everywhere on* $\partial\Omega$ *outside the discrete spectrum of the measure.*

THEOREM 3. *For arbitrary countable set of points* $\{a_k\}_{k=1}^{\infty} \subset \mathbf{C}$ *there is a measure for which*

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = 0, \qquad \textit{if} \quad z \in \{a_k\}_1^\infty,$$

and

$$\overline{\lim_{n\to\infty}}\, |q_n(z)|^{1/n} = \infty, \qquad \textit{if} \quad z \notin \{a_k\}_1^\infty.$$

1. INTRODUCTION

1. *Definitions and Objectives*

The present work is concerned with problems of asymptotic behavior of general orthogonal polynomials and their application to rational approximations.

Let μ be a finite Borel measure on $\mathbf{C}$ with compact support $S(\mu)$, and let $\mathrm{Co}(S(\mu))$ be the convex hull of $S(\mu)$. $\Omega = \Omega(\mu)$ denotes the outer domain of $S(\mu)$, i.e., the unbounded component of $\overline{\mathbf{C}}\setminus S(\mu)$, $\partial\Omega$ is its boundary. By

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$g_\Omega(z)$ we denote the Green function of the domain Ω. Let $\{q_n(z)\}_{n=0}^{\infty}$ be the orthonormal polynomials associated with μ:

$$\int q_m(z)\,\overline{q_n(z)}\,d\mu(z)=\delta_{m,n}.$$

In case of general measures it is natural to investigate so-called nth root asymptotic befavior of the orthonormal polynomials, i.e., asymptotic befavior of the sequence $\{|q_n(z)|^{1/n}\}_1^{\infty}$.

The most general results in this direction are obtaind in the monograph [StTo] of H. Stahl and V. Totik. In particular, the following theorems, which describe asymptotic behavior of the orthonormal polynomials on and outside the support of the measure, are established (see [StTo], p. 4):

THEOREM StTo-1. *For every infinite subsequence of natural numbers $\Lambda \in \mathrm{N}$ we have*

$$\varlimsup_{n\to\infty,\, n\in\Lambda} |q_n(z)|^{1/n} \geqslant e^{g_\Omega(z)},$$

for z quasi everywhere (with the exception of a subset of capacity zero) in Ω.

Besides that, if $z\in \mathrm{C}\backslash\mathrm{Co}(S(\mu))$, the following more precise estimation takes place:

$$\varliminf_{n\to\infty} |q_n(z)|^{1/n} \geqslant e^{g_\Omega(z)}$$

uniformly on compact subsets of $\mathrm{C}\backslash\mathrm{Co}(S(\mu))$.

We note here that if in the last relation we have

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = e^{g_\Omega(z)},$$

uniformly on compact subsets of $\mathrm{C}\backslash\mathrm{Co}(S(\mu))$, then the measure μ is said to be regular (see [StTo], p. 61).

THEOREM StTo-2. *For any infinite subsequence $\Lambda \in \mathrm{N}$*

$$\varlimsup_{n\to\infty,\, n\in\Lambda} |q_n(z)|^{1/n} \geqslant 1$$

for z quasi everywhere on $\partial\Omega$.

There is a broad bibliography concerning these questions in [StTo]. We only mention one of the earliest works in this direction [Ko], where, in particular, existence of a continuous singular function $\phi_1(x)$ and a jump function $\phi_2(x)$ was established, for which the corresponding measures $\phi_1(x)\,dx$ and $\phi_2(x)\,dx$ are regular.

The main question we are investigating in this paper is whether it is possible to omit the condition "quasi everywhere" in Theorems StTo-1 and StTo-2 in case of the complete sequence.

2. *Statement of Main Results*

For the case when $z \in \Omega$ we establish

THEOREM 1. *We have*

$$\overline{\lim_{n \to \infty}} |q_n(z)|^{1/n} \geqslant e^{g_\Omega(z)} \qquad \textit{everywhere in } \Omega. \tag{1}$$

This theorem has the following stronger version:

THEOREM 1′. *For arbitrary point $z_0 \in \Omega$ there exist a neighbourhood $u(z_0)$ of this point and an infinite subsequence $\Lambda(z_0)$ of natural numbers such that*

$$\underline{\lim}_{n \to \infty,\, n \in \Lambda(z_0)} |q_n(z)|^{1/n} \geqslant e^{g_\Omega(z)} \qquad \textit{uniformly in } u(z_0). \tag{1′}$$

In connection whith Theorem 1 we mention the following result:

THEOREM (see e.g. [NiSo], p. 97). *If μ is supported on the real line, then everywhere in Ω*

$$\overline{\lim_{n \to \infty}} |q_n(x)|^{1/n} > 1.$$

This theorem asserts the fulfilment of the limit relation everywhere in Ω, but in a weaker form $(1 < e^{g_\Omega(x)})$.

In the case when $z \in \partial\Omega$ the following theorem takes place

THEOREM 2. *We have*

$$\overline{\lim_{n \to \infty}} |q_n(z)|^{1/n} \geqslant 1 \tag{2}$$

everywhere on $\partial\Omega$ outside the discrete spectrum of the measure μ.

(z is called to be in the discrete spectrum of μ if $\mu(\{z\}) > 0$).

Remark. It is natural to ask by analogy with Theorem 1 whether in the right hand side of (2) we may replace 1 by $e^{g_\Omega(z)}$. ($e^{g_\Omega(z)} > 1$ for the irregular points $z \in \partial\Omega$.) The answer is negative: it is possible to show that $q_n(z)$ may grow slower then any power function $n^{1/2+\varepsilon}$, $\varepsilon > 0$, if z is a non-isolated point of $\partial\Omega$.

Theorem 2 may be strengthened in the following way:

THEOREM 2′. *We have*

$$\left[\sum_{n=0}^{\infty} |q_n^2(z)|\right]^{-1} = \mu(\{z\}) \tag{2′}$$

everywhere on $\partial\Omega$.

Relation (2′) was well-known for the case when $S(\mu) \subset \mathbf{R}$ (in this case $S(\mu) = \partial\Omega$).

Theorem 2′ yields the following

COROLLARY 1. *For every* $\varepsilon > 0$

$$\overline{\lim_{n\to\infty}}\, |q_n(z)| \cdot n^{1/2+\varepsilon} = \infty$$

everywhere on $\partial\Omega$ *outside the discrete spectrum.*

This corollary means that the sequence $q_n(z)$ cannot tend to zero faster than some power function.

The following theorem shows that in general the exceptional set in Theorem 2 cannot be reduced:

THEOREM 3. *For arbitrary countable set of points* $\{a_k\}_{k=1}^{\infty} \subset \mathbf{C}$ *a discrete measure may be constructed, concentrated at these points, such that for the corresponding orthonormal polynomials we have*

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = 0, \qquad \text{if} \quad z \in \{a_k\}_1^{\infty},$$

$$\overline{\lim_{n\to\infty}}\, |q_n(z)|^{1/n} = \infty, \qquad \text{if} \quad z \notin \{a_k\}_1^{\infty}.$$

Remark. In the both relations of Theorem 3 the convergence may be realized with arbitrary high rate (see for the exact formulation below). Moreover, the superior limit in the second relation may be replaced by ordinary one for quasi every $z \in \mathbf{C}$, i.e., we can write

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = \infty \tag{3}$$

everywhere in C with the exception of a subset E of capacity zero. (We will show below that even Hausdorf logarithmic measure of E may be equel to zero).

In spite of the fact that the exceptional set E is sufficiently small it would be disirable to guarantee fulfilment of condition (3) for some set given beforehand. In this connection the following theorem is of interest:

THEOREM 4. *For arbitrary disjoint countable sets* $\{a_k\}_1^\infty \subset \mathbf{C}$, $\{b_k\}_1^\infty \subset \mathbf{C}$, *and for arbitrary sequence of positive numbers* $\alpha_n \to 0$ *(as* $n \to \infty$*), a discrete measure, concentrated at the points* $\{a_k\}$, *can be constructed, so that for the corresponding orthonormal polynomials we have*

$$q_n(a_k) \leqslant \alpha_n, \qquad \forall n \geqslant k,$$

$$q_n(b_k) \geqslant \frac{1}{\alpha_n}, \qquad \forall n \geqslant k.$$

We note here that the estimates (1), (1′) and (2) are precise. This follows from the results of the monograph [StTo]. In particular, for regular measures we have precise equalities in relations (1) and (1′) (see [StTo], p. 60). And besides that, if the domain Ω is regular (with respect to the Dirichlet problem) than we have a precise equality in relation (2) as well (see [StTo], p. 67).

3. *Applications*

The proof of theorem 1′ is based on a theorem from [StTo] (see below) and on the following lemma which is of independent interest:

LEMMA 1. *Let* $\rho_n(z)$ *denote the distance from a point* $z \in \mathbf{C}$ *to the zeros of the polynomial* $q_n(z)$: $\rho_n(z) = \operatorname{dist}(z, \{\textit{zeros of } q_n(z)\})$. *Then*

$$\overline{\lim_{n \to \infty}}\ \rho_n(z) > 0, \qquad \forall z \in \Omega.$$

This lemma, in particular, specifies the following classical theorem (see [Sz], Theorem 6.1.1.):

Let a measure μ have a compact support $S(\mu) \subset \mathbf{R}$ and let (α, β) be an open interval such that $\mu\{(\alpha, \beta)\} > 0$. Then each polynomial $q_n(x)$ has at least one zero on (α, β) for sufficiently large indices n.

The assertion of this theorem means the following inclusion

$$x \in S(\mu) \Rightarrow \rho_n(x) \to 0.$$

Taking into account Lemma 1 we obtain

COROLLARY 2. *Let a measure* μ *have a compact support* $S(\mu) \subset \mathbf{R}$. *Then*

$$x \in S(\mu) \Leftrightarrow \rho_n(x) \to 0.$$

This corollary shows that when $S(\mu)\subset \mathrm{R}$ the support of the measure may be characterized in terms of $p_n(x)$.

Remark. The simple example of the linear Lebesque measure on the unit circle (when $q_n(z)=z^n$) shows that Corollary 2 is not true for the general case when $S(\mu)\subset \mathrm{C}$. But it will be true if we demand in addition that $\bar{\Omega}=\bar{\mathrm{C}}$ ($\bar{\Omega}$ is the closure of Ω). More generally, Corollary 2 is true in the class of the inner points of the set $\bar{\Omega}$. This assertion follows on the one hand from Lemma 1, and on the other hand from the reasoning analogous to the one in the proof of Theorem 1 from [SaTo].

Another field of application of the results is the theory of convergence of Padé approximants (in case when $S(\mu)\subset \mathrm{R}$).

COROLLARY 3. *Let $\pi_n(z)$ denote the diagonal Padé approximants to a Markov function $\hat{\mu}(z)=\int (z-t)^{-1}\,d\mu(t)$, constructed at the point $z=\infty$ (see, e.g. [BaGr]). Then*

$$\varlimsup_{n\to\infty,\, n\in\Lambda(z_0)} |\hat{\mu}(z)-\pi_n(z)|^{1/2n}\leqslant e^{-g_\Omega(z)}$$

uniformly in $u(z_0)$, where $u(z_0)$ and $\Lambda(z_0)$ are defined as in Theorem 1′.

For regular measures μ we have precise equality in the last relation.

From Corollary 3 immediately follows

COROLLARY 4. *Under the conditions of Corollary* 3 *we have*

$$\pi_n(z)\to\hat{\mu}(z)\qquad \text{uniformly in } u(z_0),\qquad z_0\notin S(\mu),$$

as $n\to\infty$, $n\in\Lambda(z_0)$.

The last result is related with an open Padé problem concerning uniform convergence of subsequences of the Padé approximants on compact subsets of the domain of analyticity of the function (see [BaGr]). The pointwise convergence of the corresponding approximants is known (see [AKW]). Corollary 4 establishes locally uniform convergence of these approximants for the Markov type functions.

2. PROOF OF THEOREM 1′

The following theorem is known (see [StTo], p. 5):

Let D be a domain, $\bar{D}\subset\Omega$. Then

$$\varliminf_{n\to\infty}\left|\frac{q_n(z)}{(z-z_1)(z-z_2)\cdots(z-z_m)}\right|^{1/n}\geqslant e^{g_\Omega(z)},$$

uniformly on compact subsets of D, where $z_1, z_2, \ldots, z_m$ are the zeros of $q_n(z)$ in D. (The zeros and their quantity depend on n).

From this theorem and Lemma 1 (wich will be proved below) follows Theorem 1′ if in the capacity of D we take some neighbourhood of the point z_0, whoose radius is less than $\overline{\lim}\, \rho_n(z_0)$.

Theorem 1′ is proved.

Proof of Corollary 3. Corollary 3 follows from Theorem 1′ if we take into account that (see, e.g. [StTo], p. 152)

$$\hat{\mu}(z) - \pi_n(z) = \frac{1}{q_n^2(z)} \int \frac{q_n^2(x)\, d\mu(x)}{z - x},$$

where $\int (z-x)^{-1} q_n^2(x)\, d\mu(x)$ is bounded on compact subsets of Ω.

Proof of Lemma 1. For a compact set, for example, for $S(\mu)$, the polynomial convex hull $Pc(S(\mu))$ is defined as following: a point z belongs to $Pc(S(\mu))$ if we have $|P(z)| \leqslant \|P\|_{\sup,\, S(\mu)}$ for every polynomial P, where $\|P\|_{\sup,\, S(\mu)}$ denotes the suppremum-norm of P on $S(\mu)$.

It is known that $Pc(S(\mu)) = \bar{\mathbb{C}} \backslash \Omega$ (see e.g. [StTo], p. 2). That means that for arbitrary point $z_0 \in \Omega$ there exists a polynomial $P_0(z)$, such that

$$\|P_0(z)\|_{\sup,\, S(\mu)} = 1, \qquad P_0(z_0) > \alpha > 1. \tag{4}$$

The following extremal property of orthonormal polynomials is known (in general for arbitrary $z_0 \in \mathbb{C}$; see [Ak], p. 80):

$$\sup_{\deg T \leqslant n} \left(T(z_0) / \|T\|_{L^2(\mu)} \right) = \sqrt{S_n}, \tag{5}$$

where $S_n = \sum_{k=0}^{n} |q_k(z_0)|^2$ and $\|T\|_{L^2(\mu)} = \left(\int |T(z)|^2\, d\mu(z)\right)^{1/2}$.

The polynomial

$$T_n(z) = \sum_{k=0}^{n} \beta_k q_k(z), \qquad \text{where} \quad \beta_k = \overline{q_k(z_0)} / \sqrt{S_n}, \tag{6}$$

provides the suppremum in this problem.

Without loss of generality we may assume that $\mu(\mathbb{C}) = 1$. Then from (4) and (5) it follows that $S_{n_0} > \alpha^2 > 1$, where $n_0 = \deg P_0$.

Next we need

Assertion 1. There is a $\gamma > 1$, such that

$$S_n > \gamma^n, \tag{7}$$

for all sufficiently large n.

Proof. Consider the polynomials $P_0^m(z), m \in \mathrm{N}$. From (4) and (5) we obtain

$$S_{n_0 m} > \alpha^{2m} = (\alpha^{2/n_0})^{n_0 m}.$$

This yields (7) for all indices n divisible by n_0, where $\gamma = \alpha^{2/n_0} > 1$. Decreasing a little γ we obtain (7) for all sufficiently large n.

Assertion 2. For every $z_0 \in \Omega$ there is a $c > 0$, $c = c(z_0)$, such that the following inequality takes place for infinitely many indices $n \in \mathrm{N}$:

$$a_n > c \cdot S_{n-1}, \tag{8}$$

where $a_n := |q_n(z_0)|^2$. ($S_n = \sum_0^n a_k$).

Proof. Assume on the contrary that for every $c > 0$ there exists a number $N = N(c)$ such that

$$a_{n+1} \leqslant c \cdot S_n, \qquad \forall n \geqslant N. \tag{9}$$

In particular $a_{N+1} \leqslant c \cdot S_N$. Consequentely:

$$S_{N+1} = a_{N+1} + S_N \leqslant cS_N + S_N = (c+1)S_N. \tag{10}$$

From (9) and (10) it follows that

$$a_{N+2} \leqslant cS_{N+1} \leqslant c(c+1)S_N. \tag{11}$$

Then (10) and (11) yield

$$S_{N+2} = a_{N+2} + S_{N+1} \leqslant c(c+1)S_N + (c+1)S_N = (c+1)^2 S_N.$$

And so on, we obtain

$$S_{N+n} \leqslant (c+1)^n S_N, \qquad \text{for all} \quad n \geqslant N. \tag{12}$$

If we take c such a small that $c + 1 < \gamma$, then (12) and (7) contradict one another for sufficiently large numbers n.

After these preliminaries we are in the position that we can prove Lemma 1. Suppose that Lemma 1 is not true, i.e. there is a $z_0 \in \Omega$, $p_n(z_0) \to 0$. From (8) it follows that

$$a_n < S_n < \frac{c+1}{c} a_n. \tag{13}$$

Let N_0 denote the set of numbers n for which (8) and (13) are true. By assumption each polynomial $q_n(z)$ has at least one zero near to z_0. Denote this zero by z_n, $z_n \to z_0$.

Consider the polynomials

$$\hat{q}_n^{\bullet}(z) = q_n(z)\,\frac{z - z'}{z - z_n},$$

where z' is a fixed point, $z' \in \Omega$, $z' \neq z_0$. We have:

$$|\hat{q}_n(z_0)/q_n(z_0)| \to \infty, \qquad n \to \infty.$$

But the ratio $\|\hat{q}_n\|_{L^2(\mu)}/\|q_n\|_{L^2(\mu)}$ remains bounded as $n \to \infty$. We claim that we can assert the same about the polynomials (see (6)) $T_n(z) = \sum_{k=0}^{n} \beta_k q_k(z)$ and $\hat{T}_n(z) = \sum_{k=0}^{n-1} \beta_k q_k(z) + \beta_n \hat{q}_n(z)$, as $n \to \infty$, $n \in N_0$.

In fact,

$$\left|\frac{\hat{T}_n(z_0)}{T_n(z_0)}\right| = \frac{|\sum_{k=0}^{n-1} \beta_k q_k(z_0) + \beta_n \hat{q}_n(z_0)|}{|\sum_{k=0}^{n} \beta_k q_k(z_0)|}$$

$$\geqslant \frac{|\beta_n \hat{q}_n(z_0)| - |\sum_0^{n-1} \beta_k q_k(z_0)|}{|\sum_0^n \beta_k q_k(z_0)|} \geqslant \left|\frac{\beta_n \hat{q}_n(z_0)}{\sum_0^n \beta_k q_k(z_0)}\right| - 1.$$

But,

$$\left|\frac{\beta_n \hat{q}_n(z_0)}{\sum_0^n \beta_k q_k(z_0)}\right| = \left|\frac{\hat{q}_n(z_0)}{q_n(z_0)}\right| \cdot \left|\frac{\beta_n q_n(z_0)}{\sum_0^n \beta_k q_k(z_0)}\right|,$$

where $|\hat{q}_n(z_0)/q_n(z_0)| \to \infty$, as $n \to \infty$, and

$$\left|\frac{\beta_n q_n(z_0)}{\sum_0^n \beta_k q_k(z_0)}\right| = \frac{a_n}{S_n} \geqslant \frac{c}{c+1},$$

as $n \in N_0$ (see (13)). Hence, $|\hat{T}_n(z_0)/T_n(z_0)| \to \infty$, as $n \to \infty$, $n \in N_0$.

Furthermore, $\|\hat{T}_n(z)\|_{L^2(\mu)}/\|T_n(z)\|_{L^2(\mu)} = \|\hat{T}_n(z)\|_{L^2(\mu)} \leqslant \|\sum_0^{n-1} \beta_k q_k(z)\|_{L^2(\mu)} + \|\beta_n \hat{q}_n(z)\|_{L^2(\mu)} = (\sum_0^{n-1} |\beta_k|^2)^{1/2} + \|\beta_n \hat{q}_n(z)\|_{L^2(\mu)} \leqslant 1 + \|\beta_n \hat{q}_n(z)\|_{L^2(\mu)} \leqslant 1 + \|\hat{q}_n(z)\|_{L^2(\mu)} = 1 + \|\hat{q}_n(z)\|_{L^2(\mu)}/\|q_n(z)\|_{L^2(\mu)}$. But $\|\hat{q}_n(z)\|_{L^2(\mu)}/\|q_n(z)\|_{L^2(\mu)}$ is bounded as $n \to \infty$. Hence $\|\hat{T}_n(z)\|_{L^2(\mu)}/\|T_n(z)\|_{L^2(\mu)}$ will also be bounded as $n \to \infty$.

From all these estimations we obtain:

$$\frac{|\hat{T}_n(z_0)|}{|T_n(z_0)|} \cdot \frac{\|T_n\|_{L^2(\mu)}}{\|\hat{T}_n\|_{L^2(\mu)}} \to \infty,$$

as $n \to \infty$, $n \in N_0$, and, in particular,

$$\frac{|\hat{T}_n(z_0)|}{\|\hat{T}_n(z)\|_{L^2(\mu)}} > \frac{|T_n(z_0)|}{\|T_n\|_{L^2(\mu)}},$$

for all sufficiently large n. But this relation contradicts the extremality of polynomials $T_n(z)$ (see (6)).

Lemma 1 is proved.

3. Proof of Theorem 2′.

Fix a point $z_0 \in \partial\Omega$. We rewrite (5) in the following form:

$$\inf_{\deg P \leqslant n} \frac{\|P\|_{L^2(\mu)}}{|P(z_0)|} = \left[\sum_{k=0}^{n} |q_k(z_0)|^2\right]^{-1/2}.$$

We shall construct a class of polynomials $P_\varepsilon(z)$ with $|P_\varepsilon(z_0)| = 1$ (or, more exactly, $|\,|P_\varepsilon(z_0)| - 1| < \varepsilon$, but that does not matter), uniformly bounded on $S(\mu)$ and such that

$$\lim_{\varepsilon \to 0} P_\varepsilon(z) = \begin{cases} 1, & z = z_0 \\ 0, & z \neq z_0,\ z \in S(\mu). \end{cases}$$

Then from Lebesgue theorem we shall have that

$$\|P_\varepsilon(z)\|^2_{L^2(\mu)} \to \mu(\{z_0\}), \qquad \text{as } \varepsilon \to 0,$$

and evidently

$$\left[\sum_{k=0}^{\infty} |q_k(z_0)|^2\right]^{-1} \leqslant \mu(\{z_0\}).$$

The inverse inequality is trivial.

For any $\varepsilon > 0$ we define a non-negative continuous function $f_\varepsilon(|z - z_0|)$, where the function $f_\varepsilon(r)$ $(r \geqslant 0)$ is defined in the following way: $f_\varepsilon(0) = 1$, $f_\varepsilon(r) = 0$, when $r \geqslant \varepsilon$, and on the interval $[0, \varepsilon]$ the function $f_\varepsilon(r)$ is defined linearly.

The following lemma takes place

Lemma (See [StTo, p. 70]). *The set $\{|P|, P$ is a polynomial$\}$ is dense in $C_+(\partial\Omega)$, i.e., every non-negative continuous function on $\partial\Omega$ can be uniformly approximated by absolute values of polynomials.*

According to this lemma there exist polynomials such that

$$|f_\varepsilon(|z - z_0|) - P_\varepsilon(z)| < \varepsilon, \qquad \text{for all} \quad z \in \partial\Omega.$$

Consequently

$$|\,|P(z_0)| - 1| < \varepsilon, \qquad |P(z)| < 1 + \varepsilon, \tag{15}$$

on $\partial\Omega$ (and according to the maximum principle on $S(\mu)$ as well). And we also have

$$|P(z)| < \varepsilon, \qquad \text{when} \quad |z - z_0| \geqslant \varepsilon, \qquad z \in \partial\Omega. \tag{16}$$

Let us prove that for the polynomials $P_\varepsilon(z)$ relation (14) is true.

If $z \in \partial\Omega$, this is evident. Let $z \in D$, where D is an open simply connected component of the set $\overline{\mathbb{C}}\setminus\Omega$. If the boundary ∂D of D is a Jordan curve, then we may apply the following (see [Go], p. 332)

THEOREM. *Let $f(z)$ be an analytic bounded function in a domain D (with a Jordan boundary ∂D). Let ∂D be divided into two parts γ_1 and γ_2. If $\overline{\lim}\, |f(z)| \leqslant M_k$, when z tends to an inner point of γ_k $(k = 1, 2)$, $z \in D$, then we have*

$$\log |f(z)| \leqslant \omega(z, \gamma_1, D) \cdot \log M_1 + \omega(z, \gamma_2, D) \cdot \log M_2, \tag{17}$$

where $\omega(z, \gamma, D)$ is the harmonic measure of the curve γ with respect to the domain D and the point z.

From (15), (16) and (17) follows (14).

This proves (14) provided the boundary of D is a Jordan curve.

Let us now consider the case when ∂D is not a Jordan curve.

Due to the uniform continuity of P_ε the inequalities (15) and (16) take place in some neighbourhood of ∂D (which depends on ε) as well. In this neighbourhood we can inscribe a closed Jordan curve ∂D_ε, which bounds a domain D_ε containing z. We may assume that for every ε the curve ∂D_ε lies through two fixed points a_1 and a_2, $a_k \in \partial D$, $a_k \neq z_0$, $k = 1, 2$.

We may now apply inequality (17) to the domain D_ε. We only have to prove that $\omega(z, \gamma_{1,\varepsilon}, D_\varepsilon)$ does not tend to zero as $\varepsilon \to 0$. Fix a point $z \in D_\varepsilon$ and intersect the domain D_ε by some simply connected Jordan domain G so that $z \in G \cap D_\varepsilon$ and $\partial G \cap \partial D_\varepsilon = \{a_1, a_2\}$. Furthermore, let $\gamma_{1,\varepsilon} = G \cap \partial D_\varepsilon$, $\gamma_{2,\varepsilon} = \partial D_\varepsilon \setminus \gamma_{1,\varepsilon}$, $\beta_2 = \partial G \cap D_\varepsilon$, $\beta_1 = \partial G \setminus \beta_2$.

Now we use the so called principle of expansion (see [Go], p. 331):

Let $\partial D = \alpha \cup \beta$. If we expand the domain D by changing only part β, then the harmonic measure $\omega(z, \alpha, D)$ increases and $\omega(z, \beta, D)$ decreases.

Applying this principle two times we obtain:

$$\omega(z, \gamma_{1,\varepsilon}, D_\varepsilon) > \omega(z, \gamma_{1,\varepsilon}, D_\varepsilon \cap G) > \omega(z, \beta_1, G).$$

Consequently, $\omega(z, \gamma_{1,\varepsilon}, D_\varepsilon)$ is bounded from below by a positive number which does not depend on ε.

Theorem 2′ is proved.

4. Proof of Theorems 3 and 4

First we prove the following two lemmas:

Lemma 2. *For arbitrary countable set $\{a_i\}_1^\infty \subset \mathbb{C}$ a finite positive measure may be constructed, concentrated at these points, for which we have*

(a) *The zeros of the corresponding orthonormal polynomials $q_n(z) = \gamma_n z^n + \cdots$ are arbitrary close to the points $\{a_i\}_1^\infty$. More exactly, if we put*

$$\delta_n = \max_{1 \leqslant i \leqslant n} \operatorname{dist}(a_i, \{\textit{zeros of } q_n(z)\}),$$

than $\delta_n \to 0$ (as $n \to \infty$) with arbitrary high rate.

(b) *$\gamma_n \to \infty$ (as $n \to \infty$) with arbitrary high rate.*

(c) *The values of $q_n(z)$ at the points $\{a_i\}_1^n$ are arbitrary small. More exactly, if we put*

$$\sigma_n = \max_{1 \leqslant i \leqslant n} |q_n(a_i)|,$$

than $\sigma_n \to 0$ (as $n \to \infty$) with arbitrary high rate.

Lemma 3. *For arbitrary countable set $\{a_k\} \subset \mathbb{C}$ there exists a sequence of positive numbers $\{r_k\}_{k=1}^\infty$ with the following property: for arbitrary point $z \notin \{a_k\}_1^\infty$ the following inequality*

$$\operatorname{dist}(z, \{a_k\}_1^n) > r_n$$

takes place for infinitly many indices $n \in \mathbb{N}$.

Proof of Lemma 2. Here we use the idea of high speed convergent series $\sum \mu_m$, wich was adopted from [StTo].

At first we prove Lemma 2 for the case when the set $\{a_i\}_1^\infty$ is bounded.

Let $Q_n(z) := q_n(z)/\gamma_n = z^n + \cdots$ denote the corresponding monic orthogonal polynomials. $\rho_n(a_m)$ is defined as in Lemma 1, i.e., $\rho_n(a_m) = \operatorname{dist}(a_m, \{\textit{zeros of } q_n(z)\})$. It is easy to verify that

$$|Q_n(a_m)| \geqslant \rho_n^n(a_m).$$

Now we shall show that the values $Q_n(a_m)$, $1 \leqslant m \leqslant n$, may tend to zero with arbitrary high rate as $n \to \infty$. Then the first assertion of Lemma 2 will follow from the last estimation above.

We construct the measure μ in the following form: at the points $a_1, a_2, \ldots$ we concentrate masses $\mu_1, \mu_2 \ldots$, where $\mu_1 = \varepsilon_1$, $\mu_2 = \varepsilon_1 \varepsilon_2$, $\mu_3 = \varepsilon_1 \varepsilon_2 \varepsilon_3, \ldots$, with $\varepsilon_i > 0$, $\sum_{i=1}^\infty \varepsilon_i \leqslant 1$.

We define the numbers $\{\varepsilon_i\}$ in consecutive order. We choose ε_1 arbitrarily $0<\varepsilon_1<1$. Now let we have defined $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$; Let us define ε_{n+1}. The monic polynonial $Q_n(z)$ has the following extremal property:

$$\sup \int |z^n + \cdots|^2 \, d\mu(z) = \int |Q_n(z)|^2 \, d\mu(z) = 1/\gamma_n^2, \tag{18}$$

where $\gamma_n > 0$ is the leading coefficient of the corresponding orthonormal polynomial $q_n(z)$.

Let $T_n(z) = (z-a_1)(z-a_2)\cdots(z-a_n)$ and let d denote the diameter of the set $\{a_k\}_1^\infty$. Then by (18) we have $\int |Q_n(z)|^2 \, d\mu(z) \leqslant \int |T_n(z)|^2 \, d\mu(z) = \sum_{k=1}^\infty |T_n(a_k)|^2 \mu_k = \Sigma_{k>n} |T_n(a_k)|^2 \mu_k \leqslant d^{2n}(\mu_{n+1} + \mu_{n+2} + \cdots) = d^{2n}\varepsilon_1\varepsilon_2 \cdots \varepsilon_{n+1}(1 + \varepsilon_{n+2} + \varepsilon_{n+3} + \cdots) \leqslant 2d^{2n}\varepsilon_1\varepsilon_2\cdots\varepsilon_{n+1}$. (Here we use that $\sum_1^\infty \varepsilon_i \leqslant 1$). Consequently,

$$\int |Q_n(z)|^2 \, d\mu(z) \leqslant 2d^{2n}\varepsilon_1\varepsilon_2\cdots\varepsilon_{n+1}. \tag{19}$$

On the other hand, we have

$$\int |Q_n(z)|^2 \, d\mu(z) \geqslant |Q_n(a_m)|^2 \cdot \mu_m, \qquad \text{for all} \quad m \geqslant 1.$$

The last two inequality together with $\mu_m = \varepsilon_1\varepsilon_2\cdots\varepsilon_m$ yield

$$|Q_n(a_m)|^2 \leqslant 2d^{2n}\varepsilon_{m+1}\varepsilon_{m+2}\cdots\varepsilon_{n+1}, \qquad \text{if} \quad m \leqslant n.$$

This estimation shows that if we choose ε_{n+1} in the proper way, all numbers $Q_n(a_m)$, $1 \leqslant m \leqslant n$, will be arbitrary small.

Besides that, from (18) and (19) it follows that γ_n is large if ε_{n+1} is small.

Assertions (a) and (b) of Lemma 2 are proved.

Furthermore, by orthonormality we have

$$1 = \int |q_n(z)|^2 \, d\mu(z) = \sum_{i=1}^{n+1} \mu_i \, |q_n(a_i)|^2 + \sum_{i=n+2}^{\infty} \mu_i \, |q_n(a_i)|^2. \tag{20}$$

if now we choose ε_{n+2} sufficiently small then the infinite sum in (20) will be arbitrary close to 0. Then the corresponding finite sum will be arbitrary close to 1. Due to assertion (a) the same fact will be true if we replace $q_n(z)$ by $\gamma_n T_n(z)$. So $\Sigma_{i=1}^{n+1} \mu_i \, |\gamma_n T_n(a_i)|^2 = \mu_{n+1} \, |\gamma_n T_n(a_{n+1})|^2$ is arbitrary close to 1. Using again assertion (a) we get that $\mu_{n+1} \, |q_n(a_{n+1})|^2$ is also arbitrary close to 1.

Now from (20) follows part (c) of Lemma 2.

Lemma 2 is proved.

Remark. We have proved Lemma 2 for the case when the set $\{a_k\}_1^\infty$ is bounded. In the general case the proof is the same. We only have to consider the masses μ_k/d_k^{2k} (d_k denotes the diameter of $\{a_i\}_{i=1}^k$) instead of μ_k beginning with the index k, for which $d_k > 1$.

Proof of Lemma 3. We define the numbers $\{r_n\}$ in the following way: we choose r_1 so that the neighbourhoods $u_{r_1}(a_1)$ and $u_{r_1}(a_2)$ are disjoint. ($u_{r_n}(a_k)$ will denote the neighbourhood of the point a_k of radius r_n.) Next, we choose r_2 $(0<r_2<r_1)$ so that the following neighbourhoods $u_{r_2}(a_1), u_{r_2}(a_2), u_{r_2}(a_3)$ are mutually disjoint. And so on, for each r_n we suppose that $0<r_n<r_{n-1}$ and

$$u_{r_n}(a_i) \cap u_{r_n}(a_j) = \varnothing, \qquad \text{for all } i, j, \qquad i<j\leqslant n+1. \tag{21}$$

Besides let $r_n \to 0$, as $n \to \infty$.

Now we prove that the assertion of Lemma 3 is true for such $\{r_n\}$. Assume on the contrary that there is a point $z \in \{a_k\}$ and a number $n_0 \in \mathrm{N}$, such that

$$z \in \bigcup_{k=1}^{n} u_{r_n}(a_k), \qquad \text{for all} \quad n \geqslant n_0. \tag{22}$$

By construction the neighbourhoods $u_{r_n}(a_k)$, $k = 1, 2, ..., n$ are mutually disjoint. Hence the point z belongs only to one of these neighbourhoods. Let $k(n)$ denote the index of this neighbourhood, i.e., let $z \in u_{r_n}(a_{k(n)})$. Because of (21) $u_{r_n}(a_{k(n)})$ and $u_{r_n}(a_{n+1})$ are disjoint too. And together with the condition: $r_{n+1}<r_n$, this implies that

$$u_{r_n}(a_{k(n)}) \cap u_{r_{n+1}}(a_k) = \varnothing, \qquad k = 1, 2, ..., n+1, \qquad k \neq k(n).$$

But according to (22) $z \in \bigcup_{k=1}^{n+1} u_{r_{n+1}}(a_k)$. So $z \in u_{r_{n+1}}(a_{k(n)})$. In other words $k(n+1)=k(n)$. If we continue this arguing we obtain that $k(n)=k(n_0)$, for all $n \geqslant n_0$. That means that the point z belongs to all neighbourhoods $u_{r_n}(a_{k(n_0)})$ of the one and the same point $a_{k(n_0)}$. But since $r_n \to 0$ we obtain: $z = a_{k(n_0)}$. But this contradicts the condition $z \notin \{a_k\}_1^\infty$.

Lemma 3 is proved.

Proof of Theorem 3. Fix a sequence of positive numbers $\alpha_n \to 0$ and numbers $\{r_n\}_1^\infty$, defined as in Lemma 3. As it follows from Lemma 2 we may construct a discrete measure concentrated at the points $\{a_k\}_1^\infty$ and such that the corresponding orthonormal polynomials have the following properties:

1. The zeros of $q_n(z)$ are arbitrary close to the points $a_1, a_2, ..., a_n$. We suppose that

$$|z_k - a_k| < r_n/2, \qquad k = 1, 2, ..., n, \tag{23}$$

where z_k is the nearest zero of $q_n(z)$ to a_k, $k = 1, 2, ..., n$.

2. The leading coefficients γ_n of $q_n(z)$ are arbitrary large.
3. The values $\{q_n(z_k)\}_{k=1}^{n}$ are arbitrary small. We suppose that

$$|q_n(a_k)| \leqslant \alpha_n, \qquad k = 1, 2, ..., n. \tag{24}$$

Due to Condition 2 we may choose γ_n such a large that

$$|q_n(z)| \geqslant 1/\alpha_n \qquad \text{for all} \quad z \notin \bigcup_{k=1}^{n} u_{r_n}(a_k) \tag{25}$$

where $u_{r_n}(a_k)$ denotes (as in Lemma 3) the neighbourhood of the point a_k of the radius r_n. That is possible because for the indicated points z we have:

$$|q_n(z)| = \gamma_n |z - z_1| \cdot |z - z_2| \cdots |z - z_n| \leqslant \gamma_n (r_n/2)^n \quad (\text{see } (23)).$$

Now the first relation of Theorem 3 follows from (24) and the second follows from (25) together with Lemma 3.

Actually, we have proved more than we assert in Theorem 3. Namely, we have proved that the convergence in the relations of the theorem may be realized with arbitrary high rate.

Let us prove, furthermore, that the limit superior in the second relation of Theorem 3 may be replaced by ordinary one for quasi every z. Denote $G_n := \bigcup_{k=1}^{n} u_{r_n}(a_k)$. From (25) it follows that if for some z the number of indices n, for wich the inclusion $z \in G_n$ takes place is finite, then for this point the limit superior may be replaced by the ordinary one.

Denote by E the set of the points which belong to infinitely many G_n. Let $E_n := \bigcup_{k \geqslant n} G_k$. It is easy to prove that $E = \bigcap_{n=1}^{\infty} E_n$.

Consider the set E_n. $E_n = \bigcup_{k \geqslant n} G_k$, where each G_k is the union of k disks of radius r_k. The conditions of Lemma 3 bound the numbers r_k only from above and therefore, we may suppose that these numbers satisfy the following condition

$$\sum_{k=1}^{\infty} k[\log(1/r_k)]^{-1} < \infty. \tag{26}$$

We need the following definition (see [La], p. 244): Let $\{u_i\}$ be a covering of a set E, where u_i is a disk of radius r_i. Suppose that $r_i < \varepsilon$ and consider the sum:

$$\sum_i [\log(1/r_i)]^{-1}.$$

Let furthermore:

$$\inf_{r_i < \varepsilon} \sum [\log(1/r_i)]^{-1} = m(E, \varepsilon),$$

where the infimum is taken from all possible covering for which $r_i < \varepsilon$. The value

$$m(E) = \lim_{\varepsilon \to 0} m(E, \varepsilon)$$

is called logarithmic measure of Hausdorff. We apply the following (see [La], p. 249):

THEOREM. *If the logarithmic measure of Hausdorff of a set E is finite than its inner logarithmic capacity equals zero.*

We apply this theorem to the set $E = \bigcap_{n \geqslant 1} E_n$. (The set E is a Borel set and in this case instead of the inner logarithmic capacity we may speak about the logarithmic capacity). Since $E \subset E_n$, $n \in \mathrm{N}$ we may consider the set E_n as a covering of the set E by disks. The following sum

$$\sum_{k \geqslant n} k[\log(1/r_k)]^{-1}$$

corresponds to the set E_n. This sum tends to zero as $n \to \infty$ (see (26)). Furthermore, from (26) it follows that $r_n \to 0$, as $n \to \infty$. Consequently, $\sup_{k \geqslant n} r_k \to 0$, as $n \to \infty$, and this implies by the definition that the logarithmic measure of Hausdorff of the set E equals zero.

Theorem 3 is proved.

Proof of Theorem 4. Just as in Theorem 3 we may construct a measure concentrated at the points $\{a_i\}$, such that:

1. The zeros of $q_n(z)$ are arbitrary close to the points $a_1, a_2, ..., a_n$.
2. The leading coefficients γ_n of $q_n(z)$ are arbitrary large.
3. The values $q_n(a_k)$, $k = 1, 2, ..., n$, are arbitrary small.

Condition 3 guarantees the following inequality:

$$|q_n(a_k)| \leqslant \alpha_n, \qquad k = 1, 2, ..., n.$$

And Conditions 1 and 2 guarantee that:

$$|q_n(b_k)| \geqslant 1/\alpha_n, \qquad k = 1, 2, ..., n. \tag{27}$$

Indeed, $|q_n(b_k)| = \gamma_n |b_k - z_1| \cdot |b_k - z_2| \cdots |b_k - z_n|$, where $z_1, z_2, ..., z_n$ are the zeros of $q_n(z)$, and they are arbitrary close to the points $a_1, a_2, ..., a_n$. Hence, we may assume that

$$\min_{1 \leq i \leq n,\, 1 \leq k \leq n} |b_k - z_i| > c_n,$$

where c_n depends only on the sets $\{a_k\}_1^n$, $\{b_k\}_1^n$. Then we have the estimation $|q_n(b_k)| \geq \gamma_n c_n^n$, where according to Condition 2, γ_n may be arbitrary large. Consequently, inequality (27) is guaranteed.

Theorem 4 is proved.

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